On the Invariance of the Sign Pattern of Matrix Eigenvectors Under Perturbation

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ABSTRACT

We study how much perturbation δA in a real matrix A is allowed for the ith real eigenvector not to change sign.

I. INTRODUCTION

The problem we investigate is, given a real matrix A with a real eigenpair (λ_i, x^i), what is the maximum allowable real perturbation δA on A which will make the components of the eigenvector $\tilde{x}^i$ of $A + \delta A$ have equal (opposite) signs to those of x^i? Alternatively, given a real interval $\lambda_i^I = [\lambda_i - \Delta\lambda_i, \lambda_i + \Delta\lambda_i]$ and a real matrix $\tilde{A} = A + \delta A$ with eigenvalue $\lambda_i + \delta\lambda_i$ lying in $\lambda_i^I(|\delta\lambda_i| \leqslant \Delta\lambda_i)$, what is the restriction on $\Delta\lambda_i$ such that $\tilde{x}^i$ has the same components signs as x^i? It is found that if $\Delta\lambda_i$ is less than one-third the separation of λ_i from the nearest λ_j, then every perturbed matrix $A + \delta A$ having an eigenvalue $\tilde{\lambda}_i$ lying in λ_i^I will keep the same sign pattern of its eigenvector.

First, we need the two following preliminary results, valid for either real or complex eigenproblems.

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II. BOUNDS FOR EIGENVALUES AND EIGENVECTORS

THEOREM 1. *All eigenvalues of $A + \delta A$ lie in the union of the disks*

$$M_i = \{z : |z - \lambda_i| \leqslant \| |T| |T^{-1} \delta A| \| \}, \qquad i = 1, \dots, n, \tag{1}$$

where $|\cdot|$ denotes the absolute value taken componentwise, $\|\cdot\|$ is some norm, and T is the modal matrix of A ($T^{-1}AT = \Lambda$), assuming A to be nondefective to have λ_i, $i = 1, \dots, n$, as eigenvalues.

Proof. From the eigenvalue problem

$$(A + \delta A)\tilde{x} = \tilde{\lambda}\tilde{x} \tag{2}$$

we get

$$\begin{aligned} \tilde{x} &= (\tilde{\lambda} I - A)^{-1} \delta A \tilde{x} \\ &= T(\tilde{\lambda} I - \Lambda)^{-1} T^{-1} \delta A \tilde{x}. \end{aligned} \tag{3}$$

Thus

$$|\tilde{x}| \leqslant \frac{1}{\min_i |\tilde{\lambda} - \lambda_i|} |T| |T^{-1} \delta A| |\tilde{x}|, \tag{4}$$

or

$$\min_i |\tilde{\lambda} - \lambda_i| \leqslant \| |T| |T^{-1} \delta A| \|. \qquad \blacksquare \tag{5}$$

Now if the disks are separated, i.e. if

$$\mathrm{sep}_i = \min_{j \neq i} |\lambda_i - \lambda_j| > 2 \| |T| |T^{-1} \delta A| \|, \tag{6}$$

then M_i contains exactly one characteristic root of $A + \delta A$, since the set M_i degenerates to the point λ_i as $\|\delta A\| \to 0$. The quantity $\| |T| |T^{-1} \delta A| \|$ therefore bounds the shift $\delta\lambda_i$. The above bound resembles the Bauer-Fike result [1] and applies to all eigenvalues real or complex.

THEOREM 2. *If λ_i is a simple eigenvalue of a nondefective matrix A with corresponding eigenvector x^i, then given a matrix δA such that* $\mathrm{sep}_i > 2\| |T| |T^{-1}\delta A| \|$, *the eigenvector of $A + \delta A$ is given by $\tilde{x}^i = x^i + \delta x^i$, where*

$$|\delta x^i| \leqslant (I - |T|D|T^{-1}\delta A|)^{-1}|T|D|T^{-1}\delta A|\,|x^i|, \tag{7}$$

in which

$$D = \mathrm{diag}\left(\frac{1}{|\lambda_1 - \lambda_i| - |\delta\lambda_i|}, \dots, 0_i, \dots, \frac{1}{|\lambda_n - \lambda_i| - |\delta\lambda_i|}\right), \tag{8}$$

where $\delta\lambda_i$ is the shift in the eigenvalue λ_i ($|\delta\lambda_i| \leqslant \| |T| |T^{-1}\delta A| \|$ *as a result of Theorem* 1).

Proof. From the eigenvalue problem

$$(A + \delta A)(x^i + \delta x^i) = (\lambda_i + \delta\lambda_i)(x^i + \delta x^i) \tag{9}$$

we have

$$A\,\delta x^i - \lambda_i\,\delta x^i = -\delta A\tilde{x}^i + \delta\lambda_i\,x^i + \delta\lambda_i\,\delta x^i. \tag{10}$$

Set

$$\delta x^i = \sum_{j \neq i} \alpha_{ji} x^j \tag{11}$$

with $\alpha_{ii} = 0$ to avoid renormalization of $\tilde{x}^i$. Since by writing $\tilde{x}^i = x^i + \sum_{j=1}^n \beta_{ji} x^j = (1 + \beta_{ii})x^i + \sum_{j \neq i} \beta_{ji} x^j$, we can divide $\tilde{x}^i$ by $1 + \beta_{ii}$ [$1 + \beta_{ii} \neq 0$; otherwise, from (9), taking $\tilde{x}^i = \sum_{j \neq i} \beta_{ji} x^j$, we can arrive at a result contradicting the fact that $\mathrm{sep}_i > 2\| |T| |T^{-1}\delta A| \|$] to yield an eigenvector deviating from x^i by the amount in (11). Substituting from (11) into (10) and premultiplying the latter by $(y^k)^T$ (an eigenrow of A), $k \neq i$, yields

$$(\lambda_k - \lambda_i)a_{ki} = -(y^k)^T\,\delta A\tilde{x}^i + \delta\lambda_i\,\alpha_{ki}, \qquad k \neq i, \tag{12}$$

giving

$$(|\lambda_k - \lambda_i| - |\delta\lambda_i|)|\alpha_{ki}| \leqslant \left|(y^k)^T \delta A\right||\tilde{x}^i|, \qquad k \neq i. \tag{13}$$

But $|\delta\lambda_i| < \frac{1}{2}\,\mathrm{sep}_i \leqslant \frac{1}{2}|\lambda_k - \lambda_i|$, $k \neq i$, so in vector form

$$|\alpha_i| \leqslant D|T^{-1}\,\delta A|\,|\tilde{x}^i|. \tag{14}$$

Thus from (11)

$$|\delta x^i| \leqslant |T|\,|\alpha_i| \leqslant |T|D|T^{-1}\,\delta A|\,|\tilde{x}^i|. \tag{15}$$

Since $\tilde{x}^i = x^i + \delta x^i$, we reach

$$(I - |T|D|T^{-1}\,\delta A|)|\delta x^i| \leqslant |T|D|T^{-1}\,\delta A|\,|x^i|. \tag{16}$$

But the fact that

$$\begin{aligned}\mathrm{sep}_i = \min_{k \neq i}|\lambda_k - \lambda_i| &> 2\big\||T|\,|T^{-1}\,\delta A|\big\|\\ &\geqslant \big\||T|\,|T^{-1}\,\delta A|\big\| + |\delta\lambda_i|,\end{aligned} \tag{17}$$

i.e. that

$$\frac{\big\||T|\,|T^{-1}\,\delta A|\big\|}{\mathrm{sep}_i - |\delta\lambda_i|} < 1, \tag{18}$$

entails that

$$\begin{aligned}\rho(|T|D|T^{-1}\,\delta A|) &\leqslant \big\||T|D|T^{-1}\,\delta A|\big\|\\ &\leqslant \frac{\big\||T|\,|T^{-1}\,\delta A|\big\|}{\mathrm{sep}_i - |\delta\lambda_i|}\\ &< 1,\end{aligned}$$

from which (7) finally follows in view of (16). ■

III. SIGN INVARIANCE

Now, we turn our attention to the particular situation in which A and δA are real and A has a simple real eigenvalue λ_i.

Before anything is said concerning the sign pattern of the eigenvector $\tilde{x}^i$ of $A + \delta A$ in comparison with that of x^i, we ask ourselves whether $\tilde{x}^i$ is ever real. For this, we have

LEMMA. *If λ_i is a real simple eigenvalue of a real nondefective matrix A, then under a real perturbation δA such that* $\mathrm{sep}_i > 2\| |T| |T^{-1}\delta A| \|$, *the eigenvector $\tilde{x}^i$ of $A + \delta A$ is real.*

Proof. $\tilde{\lambda}_i$ lies in a separate disk of center λ_i as a result of Theorem 1. If $\tilde{\lambda}_i$ is to be complex, then its conjugate value must exist as an eigenvalue, since $A + \delta A$ is real. It must also lie in the same disk. But λ_i is simple and cannot be split by perturbation into two eigenvalues. Thus $\tilde{\lambda}_i$ is real, and the real equation $(A + \delta A - \tilde{\lambda}_i I)\,\tilde{x}^i = 0$ has a nontrivial real solution $\tilde{x}^i$. ■

Despite the fact that $\tilde{x}^i$ is real, no reason has yet been given why it should keep the same sign pattern as x^i. It does, however, if

$$|\delta x^i| < |x^i|, \tag{19}$$

and we have

THEOREM 3. *If λ_i is a real simple eigenvalue of a real nondefective matrix A with modal matrix T and corresponding to an eigenvector x^i, then any real perturbation δA such that* $\mathrm{sep}_i > 2\| |T| |T^{-1}\delta A| \|$ *and satisfying*

$$\left(I - |T|D|T^{-1}\delta A| \right)^{-1} |T|D|T^{-1}\delta A|\,|x^i| < |x^i| \tag{20}$$

ensures that the eigenvector $\tilde{x}^i$ of $A + \delta A$ will have same sign pattern as x^i.

The proof follows from (7) and (20), from which (19) is true.

Finally, instead of imposing a restriction on δA as in (20) for $|\delta x^i| < |x^i|$, we search for a condition on $|\delta\lambda_i|$ itself. Thus, by defining a real interval $\lambda_i^I = [\lambda_i - \Delta\lambda_i, \lambda_i + \Delta\lambda_i]$ enclosing $\tilde{\lambda}_i$, we give a bound for $\Delta\lambda_i$ such that $|\delta x^i| < |x^i|$. First, we have

LEMMA. *If* $\mathrm{sep}_i > 2\| |T| |T^{-1}\delta A| \|$, *and there is a real interval* $\lambda_i^I = [\lambda_i - \Delta\lambda_i, \lambda_i + \Delta\lambda_i]$ *such that*

$$|T| |T^{-1}\delta A| |\tilde{x}^i| \leqslant \Delta\lambda_i |\tilde{x}^i|, \tag{21}$$

then

$$\tilde{\lambda}_i \in \lambda_i^I \qquad (|\delta\lambda_i| \leqslant \Delta\lambda_i) \tag{22}$$

The proof follows directly from (4). Finally,

THEOREM 4. *If (λ_i, x^i) is a real eigenpair of a real nondefective matrix A with λ_i simple, and given a real interval $\lambda_i^I = [\lambda_i - \Delta\lambda_i, \lambda_i + \Delta\lambda_i]$, then if*

$$\text{sep}_i > 3\,\Delta\lambda_i, \tag{23}$$

then all real matrices $A + \delta A$, such that $\text{sep}_i > 2\| |T|\,|T^{-1}\,\delta A| \|$ and for which $|T|\,|T^{-1}\,\delta A|\,|\tilde{x}^i| \leqslant \Delta\lambda_i |\tilde{x}^i|$, have an eigenpair $(\tilde{\lambda}_i, \tilde{x}^i)$ with $\tilde{\lambda}_i$ lying in λ_i^I and the eigenvector $\tilde{x}^i$ having same sign pattern as x^i.

Proof. From (15) and (21)

$$|\delta x^i| \leqslant \frac{\Delta\lambda_i}{\text{sep}_i - \Delta\lambda_i}|\tilde{x}^i|. \tag{24}$$

Since $\tilde{x}^i = x^i + \delta x^i$,

$$|\delta x^i| \leqslant \left(1 - \frac{\Delta\lambda_i}{\text{sep}_i - \Delta\lambda_i}\right)^{-1} \frac{\Delta\lambda_i}{\text{sep}_i - \Delta\lambda_i}|x^i| \tag{25}$$

$$= \frac{\Delta\lambda_i}{\text{sep}_i - 2\,\Delta\lambda_i}|x^i|. \tag{26}$$

So if $\Delta\lambda_i/(\text{sep}_i - 2\,\Delta\lambda_i) < 1$, i.e. if $\text{sep}_i > 3\,\Delta\lambda_i$, then $|\delta x^i| < |x^i|$. ∎

EXAMPLE.

$$A = \begin{pmatrix} 6 & 8 \\ 8 & -6 \end{pmatrix}, \qquad \delta A = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix},$$

$$\lambda_1 = 10, \qquad \lambda_2 = -10,$$

$$T = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \qquad T^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix},$$

and

$$|T|\,|T^{-1}\,\delta A| = \frac{1}{5}\begin{pmatrix} 5\alpha & 0 \\ 4\alpha - 3\beta & 0 \end{pmatrix}$$

for

$$2\alpha + \beta \geqslant 0, \qquad \alpha - 2\beta \geqslant 0. \tag{i}$$

Investigating the situation at λ_1, we have

$$|\delta\lambda_1| \leqslant \big\| |T|\,|T^{-1}\,\delta A| \big\|_1 = \tfrac{1}{5}(9\alpha - 3\beta), \tag{ii}$$

and from sep $> 2\| \, |T|\,|T^{-1}\,\delta A| \, \|$ we get

$$50 > 9\alpha - 3\beta. \tag{iii}$$

Further, upon invoking $|T|\,|T^{-1}\,\delta A|\,|\tilde{x}^1| \leqslant \Delta\lambda_1 |\tilde{x}^1|$, with $\Delta\lambda_1$ bounded by (ii), and

$$\tilde{x}^1 = \left(16, \sqrt{\alpha^2 + 400 + 24\alpha + 32\beta} - 12 - \alpha\right)^T,$$

we obtain a cubic relation in α and β in the form

$$f(\alpha, \beta) \leqslant 0. \tag{iiii}$$

Plotting (i), (iii), and (iiii) suggests the choice of the extreme point $\alpha = 6.666\ldots$, $\beta = 3.333\ldots$ to obtain $\tilde{\lambda}_1 = 16.666\ldots$ with $\delta\lambda_1 = 6.666\ldots = \frac{1}{3}|\lambda_1 - \lambda_2|$ and an associated eigenvector $\tilde{x}^1$ having same sign as x^1.

On the contrary, with the choice of $\alpha = 10.67$ and $\beta = -8$ yet $\tilde{\lambda}_1 = 16.67$, $\tilde{x}^1$ starts to have a different sign than x^1. The reason is that (iii) and (iiii) are violated. Similar results can be obtained if α and β are chosen so as to make $\tilde{\lambda}_1$ lie to the left of λ_1.

IV. CONCLUSION

The invariance of the sign pattern of matrix eigenvectors under perturbation, apart from its possible immediate interest to some readers, has direct applications in interval eigenvalue problems [2–4]. It has been seen that the bounds on exact eigenvalue ranges can be easily found once the eigenvectors keep their components signs invariant. The difference between Theorems 3 and 4 is that the former applies to the situation in which an interval matrix is given and it is required to check the invariance of the sign pattern of the eigenvectors before proceeding to compute the interval eigenvalue ranges, whereas the latter becomes important if the eigenvalue ranges are given and it is required to compute interval entries of A. The condition of sign pattern invariance must therefore be applied only on given eigenvalue ranges which qualify as eigenvalues of some interval matrices having invariant sign patterns.

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